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On Well-Posedness of the Second Order Accuracy Difference Scheme for Reverse Parabolic Equations

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ABSTRACT

The second order accuracy difference scheme for the approximate solution of the abstract reverse parabolic equation in a Hilbert space with the nonlocal boundary condition is considered. The stability estimates, almost coercivity, and coercivity estimates for the solution of this difference scheme are established. New coercivity inequalities for the solution of multipoint nonlocal boundary value difference equations of reverse parabolic type are obtained.

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1. INTRODUCTION

The importance of the coercivity inequalities (well-posedness) in the study of boundary value problems for partial differential equations is well known (see Ladyzhenskaya *et al.* (1967); Ladyzhenskaya *et al.* (1968) and Vishik *et al.* (1959)). Many researchers have been extensively studied the coercivity inequalities for nonlocal boundary value problems for parabolic partial differential equations (see the references therein).

In Ashyralyev *et al.* (2006), the multipoint nonlocal boundary value problem

$$\begin{cases} u'(t) - Au(t) = f(t) \quad (0 \le t \le 1), \\ u(1) = \sum_{k=1}^{p} \alpha_k u(\theta_k) + \varphi, \\ 0 \le \theta_1 < \theta_2 < \cdots < \theta_p < 1 \end{cases} \tag{1}$$

in a Hilbert space H with self-adjoint positive definite operator A is considered under assumption

$$\sum_{k=1}^{p}|\alpha_k|\leq 1. \tag{2}$$

We established the well-posedness of multipoint nonlocal boundary value problem (1). New coercivity estimates in various Hölder norms for the solutions of nonlocal boundary value problems for parabolic equations are obtained.

In Ashyralyev *et al.* (2009), we studied the Rothe difference scheme for the approximate solution of abstract parabolic equation (1). We established the stability estimates, almost coercivity and coercivity estimates for the solution of this difference scheme. In applications, we obtained new coercivity inequalities for the solution of multipoint nonlocal boundary value difference equations of parabolic type.

Throughout the present paper, we let M denote positive constants, which may differ time to time and is not a subject of precision. However, $M(\alpha,\beta,\ldots)$ is used to focus on the fact that the constant depends only on $\alpha,\beta,\ldots$

Let $[0,1]_\tau=\{t_k=k\tau,\ k=\overline{1,N},\ N\tau=1\}$ denote the *uniform grid space* with step size $\tau>0$, where N is a fixed positive integer. Let $F_\tau(H)=F([0,1]_\tau,H)$ be the linear space of grid functions $\varphi^\tau=\{\varphi_k\}_1^N$ with values in the Hilbert space H. Denote the Banach space of bounded grid functions by $C_\tau(H)=C([0,1]_\tau,H)$ with norm

$$\left\|\varphi^\tau\right\|_{C_\tau(H)}=\max_{1\leq k\leq N}\left\|\varphi_k\right\|_H.$$

For $\alpha\in[0,1]$, let $C^\alpha(H)=C^\alpha([0,1]_\tau,H)$ and $C_1^\alpha(H)=C_1^\alpha([0,1]_\tau H)$ be respectively the Hölder space with the norms

$$\left\|\varphi^\tau\right\|_{C^\alpha(H)}=\left\|\varphi^\tau\right\|_{C_\tau(H)}+\max_{1\leq k<k+r\leq N}\frac{\left\|\varphi_{k+r}-\varphi_k\right\|_H}{(r\tau)^\alpha},$$

$$\left\|\varphi^{\tau}\right\|_{C_1^{\alpha}(H)}=\left\|\varphi^{\tau}\right\|_{C_{\tau}(H)}+\max_{1\le k<k+r\le N}\frac{((N-k)\tau)^{\alpha}\left\|\varphi_{k+r}-\varphi_k\right\|_H}{(r\tau)^{\alpha}}.$$

In the present paper, the second order of accuracy difference scheme

$$\begin{cases}\tau^{-1}(u_k-u_{k-1})-A\left(I+\frac{\tau A}{2}\right)u_{k-1}=\psi_k,\\ \psi_k=\left(I+\frac{\tau A}{2}\right)f(t_{k-\tau/2}),\ t_k=k_{\tau},\ 1\le k\le N,\ N_{\tau}=1,\\ u_N=\sum_{m=1}^{p}\alpha_m\left\{\left(I+d_mA\right)u_{\ell m}+d_m\left(I+\frac{\tau A}{2}\right)^{-1}\psi_{\ell m+1}\right\}+\varphi,\\ d_m=\theta_m-[\theta_m/\tau]\tau,\ \ell_m=[\theta_m/\tau],\ m=\overline{1,p}\end{cases}\tag{3}$$

for approximately solving problem (1) studied. The well-posedness of multipoint nonlocal boundary value problems (3) in spaces $C_1^{\alpha}(H)$ and $C^{\alpha}(H)$ is established. Moreover, by applying these abstract results, we obtain new coercivity estimates in various Hölder norms for the solutions of nonlocal boundary value problems for parabolic equations.

2. WELL-POSEDNESS

Throughout the paper, H is a Hilbert space and A is a positive definite self-adjoint operator with $A\ge\delta I$ for some $\delta>0$, where I is the identity operator. Moreover, let $B=I+(\tau A)/2$ and $D=\left(I+\tau A+(\tau A)^2/2\right)^{-1}$.

Lemma 1. (Ashyralyev and Sobolevskii (1994)). The following estimates hold:

$$\left\|\left(I+\tau A\right)D\right\|_{H\to H}\le 1,\ \left\|BD\right\|_{H\to H}\le 1,\ \left\|B^{-1}\right\|_{H\to H}\le 1\tag{4}$$

$$\left\|D^m-e^{-m\tau A}\right\|_{H\to H}\le M\tau^2(m\tau)^{-2},\ m\ge 1,\tag{5}$$

$$\left\|(\tau A)^{\alpha}D\right\|_{H\to H}\le 1,\ \alpha=\overline{0,2},\left\|(\tau A)^{\alpha}DB\right\|_{H\to H}\le 1,\ \alpha=\overline{0,1},\tag{6}$$

$$\left\|(\tau A)^{\alpha}D(I+\tau A)\right\|_{H\to H}\le 3,\ \alpha=\overline{0,1},\tag{7}$$

$$\left\|(I+\tau A)^{\alpha} DB\right\|_{H\to H} \le 2\left\|BDB\right\|_{H\to H} \le 1, \quad (8)$$

$$\left\|(\tau A)^{\beta} D^{m}\right\|_{H\to H} \le m^{-\beta}, \ m\ge 1, \ 0\le \beta \le 1, \quad (9)$$

$$\left\|A^{\beta}(D^{m+r}-D^{m})\right\|_{H\to H} \le M(r\tau)^{\gamma}(m\tau)^{-\beta-\gamma},$$
$$1\le m< m+r\le N, \ 0\le\beta, \ \gamma\le 1 \quad (10)$$

for some M, $\delta>0$ independent of τ, where τ is a positive small number.

Lemma 2. If (2) holds, then the operator $I-\sum_{k=1}^{p}\alpha_k\left(I+d_kA\right)D^{N-[\theta_k/\tau]}$ has an inverse T_τ and the following estimate is satisfied

$$\left\|T_\tau\right\|_{H\to H} \le M(\delta,\theta_p). \quad (11)$$

Now, we obtain the formula for the solution of problem (12). Clearly, the second order of a accuracy difference scheme

$$\begin{cases} \tau^{-1}(u_k-u_{k-1})-ABu_{k-1}=\psi_k, \\ \psi_k = Bf(t_{k-\tau/2}), \ t_k=k_\tau, \ 1\le k\le N, \ N_\tau=1, \\ u_N=\sum_{m=1}^{p}\alpha_m\left\{\left(I+d_mA\right)u_{\ell m}+d_mB^{-1}\psi_{\ell_m+1}\right\}+\varphi, \\ d_m=\theta_m-[\theta_m/\tau]\tau, \ \ell_m=[\theta_m/\tau], \ m=\overline{1,p} \end{cases} \quad (12)$$

has a solution and the following formula holds

$$u_{k-1}=D^{N-k+1}u_N-\sum_{j=k}^{N}D^{j-k+1}\psi_j\tau, \ k=\overline{1,N}. \quad (13)$$

By formula (13), the nonlocal boundary condition and Lemma 2, we obtain

$$u_N=T_\tau\left\{-\sum_{m=1}^{p}\sum_{j=\ell_m+1}^{N}\alpha_m(I+d_mA)D^{j-\ell_m}\psi_j\tau+\sum_{m=1}^{p}\alpha_m d_m B^{-1}\psi_{\ell_m+1}+\varphi\right\}. \quad (14)$$

Thus, difference equation (12) is uniquely solvable and solutions satisfy (13) and (14).

Difference problem (3) is said to be stable in $F_\tau(H)$, if the stability estimate holds

$$\left\|\{u_k\}_1^N\right\|_{F_\tau(H)} \le M\left(\left\|\varphi^\tau\right\|_{F_\tau(H)} + \left\|\varphi^\tau\right\|_H\right).$$

We say problem (3) is well-posed in $F_\tau(H)$, if for every $\varphi^\tau \in F_\tau(H)$ problem (3) is uniquely solvable and also the following coercivity estimate is valid

$$\left\|\{\tau^{-1}(u_k - u_{k-1})\}_1^N\right\|_{F_\tau(H)} + \left\|\left\{A\left(I + \frac{\tau A}{2}\right)u_{k-1}\right\}_1^N\right\|_{F_\tau(H)}$$

$$\le M\left(\left\|\varphi^\tau\right\|_{F_\tau(H)} + \left\|A\varphi^\tau\right\|_{H'}\right),$$

where $H' \subset H$.

Theorem 3. Let assumption (2) hold and $\varphi \in H$. Then, the solution of difference scheme (12) satisfy the following stability estimate

$$\left\|\{u_k\}_0^N\right\|_{C_\tau(H)} \le M\left(\delta, \theta_p\right)\left(\left\|\varphi\right\|_H + \left\|\psi^\tau\right\|_{C_\tau(H)}\right). \tag{15}$$

Proof. By formula (13), estimate (6) for $\alpha = 0,$ and $N_\tau = 1,$ we get for $k = \overline{0, N-1}$

$$\left\|u_k\right\|_H \le \left\|u_N\right\|_H + \left\|\psi^\tau\right\|_{C_\tau(H)}.$$

Estimate (15) follows from (14), assumption (2), estimates (4), (6) for $\alpha = 0,$ (11) and $N_\tau = 1.$ This is the end of Theorem 3.

Theorem 4. Let $\varphi \in D(A)$. Then, the solution of difference problem (12) satisfies the following almost coercivity inequality

$$\left\|\{\tau^{-1}(u_k - u_{k-1})\}_1^N\right\|_{C_\tau(H)} + \left\|\{ABu_{k-1}\}_1^N\right\|_{C_\tau(H)}$$

$$\leq M\left(\delta,\theta_p\right)\left(\min\left\{\ln\frac{1}{\tau},1+\ln\|A\|_{H\to H}\right\}\|\psi^\tau\|_{C_\tau(H)}+\|A\varphi\|_H\right). \quad (16)$$

Proof. From formula (13), estimate (4), (6) for $\alpha=0$ it follows for $k=\overline{1,N}$

$$\|BAu_{k-1}\|_H\leq\|Au_N\|_H+\|\psi^\tau\|_{C_\tau(H)}\sum_{j=k}^{N}\left\|\tau AD^{j-k}DB\right\|_{H\to H}.$$

By using the estimates (6) for $\alpha=0$, (10) for $\gamma=1,\ \beta=0,$ we get

$$\sum_{j=k}^{N}\left\|\tau AD^{j-k}DB\right\|_{H\to H}=\|\tau ADB\|_{H\to H}+\sum_{m=1}^{N-k}\tau\left\|AD^mDB\right\|_{H\to H}$$

$$\leq\min\left\{\ln\frac{1}{\tau},1+\ln\|A\|_{H\to H}\right\}. \quad (17)$$

It follows from formula (14), assumption (2), estimates (4), (6) for $\alpha=0$, (11) and $N\tau=1$ that

$$\|Au_N\|_H\leq M(\delta,\theta_p)\left(\|\psi^\tau\|_{C_\tau(H)}\min\left\{\ln\frac{1}{\tau},1+\ln\|A\|_{H\to H}\right\}+\|A\varphi\|_H\right). \quad (18)$$

Hence, combining estimates (17), (18), we get

$$\left\|\{ABu_{k-1}\}_1^N\right\|_{C_\tau(H)}\leq M(\delta,\theta_p)$$

$$\times\left(\|\psi^\tau\|_{C_\tau(H)}\min\left\{\ln\frac{1}{\tau},1+\ln\|A\|_{H\to H}\right\}+\|A\varphi\|_H\right). \quad (19)$$

Estimate (16) follows from difference equation (12), the triangle inequality and estimate (19). This concludes the proof of Theorem 4.

Theorem 5. Assume that (2) holds and $\varphi \in D(A)$. Then, the solution of difference problem scheme (12) satisfy the following stability estimate

$$\left\|\{\tau^{-1}(u_k - u_{k-1})\}_1^N\right\|_{C_\tau^\alpha} + \left\|\{ABu_{k-1}\}_1^N\right\|_{C_1^\alpha(H)} \le M\left(\delta, \theta_p\right)\left(\frac{\left\|\psi^\tau\right\|_{C_1^\alpha(H)}}{\alpha(1-\alpha)} + \left\|A\varphi\right\|_H\right). \quad (20)$$

Proof. Using formula (13) and identity

$$\tau ADB = I - D, \quad (21)$$

we get for $k = \overline{1, N}$ that

$$ABu_{k-1} = BD^{N-k+1}Au_N - \sum_{j=k}^{N} \tau AD^{j-k}DB(\psi_j - \psi_k) + (D^{N-k+1} - I)\psi_k. \quad (22)$$

Hence, it follows from estimates (6) for $\alpha = 0,1$ (9) for $\beta = 1$ and the definition of $C_1^\alpha(H)$-norm that

$$\left\|ABu_{k-1}\right\|_H \le \left\|Au_N\right\|_H + \frac{4}{\alpha}\left\|\psi^\tau\right\|_{C_1^\alpha(H)}. \quad (23)$$

Next, let us estimate $\left\|Au_N\right\|_H$. Using formula (14) and identity (21), we obtain

$$Au_N = T_\tau \left\{ -\sum_{k=1}^{p} \alpha_k (I + d_k A) \sum_{j=\ell_k+1}^{N} \tau AD^{j-\ell k}(\psi_j - \psi_{\ell_k+1}) - \sum_{k=1}^{p} \alpha_k \beta^{-1} \psi_{\ell_k+1} + \sum_{k=1}^{p} \alpha_k (I + d_k A) D^{N-\ell_k} B^{-1} \psi_{\ell_k+1} + A\varphi \right\}.$$

Thus, from estimates (4), (7) for $\alpha = 0$, (9) for $\beta = 1$, (11), the definition of $C_1^{\alpha}(H)$-norm and assumption (2) it follows that

$$\|Au_N\|_H \leq M(\delta, \theta_p)\left(\frac{1}{\alpha}\|\psi^{\tau}\|_{C_1^{\alpha}(H)} + \|A\varphi\|_H\right). \tag{24}$$

By estimates (23), (24), we get

$$\left\|\{ABu_{k-1}\}_1^N\right\|_{C_{\tau}(H)} \leq M(\delta, \theta_p)\left(\frac{1}{\alpha}\|\psi^{\tau}\|_{C_1^{\alpha}(H)} + \|A\varphi\|_H\right). \tag{25}$$

We now estimate

$$\max_{1 \leq k < k+r \leq N} \frac{((N-k+1)\tau)^{\alpha}\|ABu_{k-1+r} - ABu_{k-1}\|_H}{(r\tau)^{\alpha}}.$$

Let $N - k + 1 > 2r$. It follows from formula (13) that

$$ABu_{k-1} - ABu_{k-1+r} = B(D^{N-k+1} - D^{N-k-r+1})Au_N \tag{27}$$

$$- \sum_{j=k}^{k+2r-1} \tau AD^{j-k}DB(\psi_j - \psi_k) - \sum_{j=k+2r}^{N} \tau A(D^{j-k}D^{j-k-r})DB(\psi_j - \psi_k)$$

$$+ \sum_{j=k+r}^{k+2r-1} \tau AD^{j-(k+r)}DB(\psi_j - \psi_{k+r}) + (I - D^{r-1})(\psi_{k+r} - \psi_k)$$

$$+(D^{N-k+1} - D^{N+1-k-r})\psi_k = P_1(k) + P_2(k) + P_3(k) + P_4(k) + P_5(k) + P_6(k).$$

Let us start with estimate $P_1(k)$. By using estimates (4), (10) for $\beta = 0$, (24) and the fact $N - k + 1 > 2r$, we obtain

$$\|P_1(k)\|_H \leq M(\delta, \theta_p)\frac{(r\tau)^{\alpha}}{((N-k+1)\tau)^{\alpha}}\left(\frac{1}{\alpha}\|\psi^{\tau}\|_{C_1^{\alpha}(H)} + \|A\varphi\|_H\right). \tag{28}$$

From estimates (6) for $\alpha=0$, (9) for $\beta=0$, the definition of $C_1^{\alpha}(H)$-norm and the fact $N-k+1>2r$, it follows that

$$\|P_2(k)\|_H \le \frac{(r\tau)^{\alpha}}{((N-k+1)\tau)^{\alpha}}\frac{4^{\alpha}}{\alpha}\|\psi^{\tau}\|_{C_1^{\alpha}(H)}. \tag{29}$$

Using estimates (6) for $\alpha=0$, (10) for $\beta=1$, $\gamma=1$, the definition of $C_1^{\alpha}(H)$-norm, the facts $N-k+1>2r$ and $j-k\ge 2r$, we get

$$\|P_3(k)\|_H \le \frac{M4^{\alpha}}{1-\alpha}\frac{(r\tau)^{\alpha}}{((N-k+1)\tau)^{\alpha}}\|\psi^{\tau}\|_{C_1^{\alpha}(H)}. \tag{30}$$

Estimates (6) for $\alpha=0$, (9) for $\beta=1$, the definition of $C_1^{\alpha}(H)$-norm and the facts $N-k+1>2r$ result that

$$\|P_4(k)\|_H \le \frac{3^{\alpha}}{\alpha}\frac{(r\tau)^{\alpha}}{((N-k+1)\tau)^{\alpha}}\|\psi^{\tau}\|_{C_1^{\alpha}(H)}. \tag{31}$$

It follows from estimates (6) for $\alpha=0$ and the definition of $C_1^{\alpha}(H)$-norm that

$$\|P_5(k)\|_H \le 2^{1+\alpha}\frac{(r\tau)^{\alpha}}{((N-k+1)\tau)^{\alpha}}\|\psi^{\tau}\|_{C_1^{\alpha}(H)}. \tag{32}$$

By estimate (10) for $\beta=0$, $\gamma=\alpha$, and the fact $N-k+1>2r$, we obtain

$$\|P_6(k)\|_H \le 3^{\alpha}\frac{(r\tau)^{\alpha}}{((N-k+1)\tau)^{\alpha}}\|\psi^{\tau}\|_{C_1^{\alpha}(H)}. \tag{33}$$

Combining estimates (28)-(33), for $N-k+1>2r$ we get

$$\frac{((N-k+1)\tau)^{\alpha}\|ABu_{k-1+r}-ABu_{k-1}\|_H}{(r\tau)^{\alpha}} \le M(\delta,\theta_p)\left(\frac{\|\psi^{\tau}\|_{C_1^{\alpha}(H)}}{\alpha(1-\alpha)}+\|A\varphi\|_H\right). \tag{34}$$

From estimates (26) and (34) it follows that

$$\max_{1\le k<k+r\le N} \frac{((N-k+1)\tau)^{\alpha} \left\|ABu_{k-1+r} - ABu_{k-1}\right\|_H}{(r\tau)^{\alpha}}$$

$$\le M(\delta,\theta_p)\left(\frac{1}{\alpha(1-\alpha)}\left\|\psi^{\tau}\right\|_{C_1^{\alpha}(H)} + \left\|A\varphi\right\|_H\right). \tag{35}$$

Thus, combining estimates (25), (35), we obtain

$$\left\|\left\{ABu_{k-1}\right\}_1^N\right\|_{C_1^{\alpha}(H)} \le M(\delta,\theta_p)\left(\frac{1}{\alpha(1-\alpha)}\left\|\psi^{\tau}\right\|_{C_1^{\alpha}(H)} + \left\|A\varphi\right\|_H\right). \tag{36}$$

Therefore, we obtain estimate (20) by using difference equation (12), estimate (36) and the triangle inequality. This is the end of Theorem5.

Let $H'_{\alpha} = H'_{\alpha,\infty}(H,A)$ denote the fractional space, consisting of all $\upsilon \in H$ for which the norm

$$\left\|\upsilon\right\|_{H'_{\alpha}} = \left\|\upsilon\right\|_H + \left\|A^{\alpha}\upsilon\right\|_H$$

is finite.
Recall that (see Ashyralyev and Sabolevskii (1994))

$$A^{\alpha}\upsilon = A^{\alpha-1}A\upsilon = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^{\infty} s^{\alpha-1}(s+A)^{-1}A\upsilon ds. \tag{37}$$

Theorem 6. Suppose $\psi^{\tau} \in C^{\alpha}(H)$, $\psi_N - \sum_{k=1}^{p}\alpha_k\psi_{\ell_k+1} + A\varphi \in H'_{\alpha}$ and (2). Then, problem (12) is well-posed in $C^{\alpha}(H)$ and the coercivity estimate holds

$$\left\|\{\tau^{-1}(u_k - u_{k-1})\}_1^N\right\|_{C^{\alpha}(H)} + \left\|\left\{ABu_{k-1}\right\}_1^N\right\|_{C_{\tau}^{\alpha}(H)} + \left\|\{\tau^{-1}(u_k - u_{k-1})\}_1^N\right\|_{C_{\tau}(H'_{\alpha})}$$

$$\le M\left(\left\|\psi_N - \sum_{k=1}^{p}\alpha_k\psi_{\ell_k+1} + A\varphi\right\|_{H'_{\alpha}} + \frac{M(\delta,\theta_p)}{\alpha^2(1-\alpha)}\left\|\psi^{\tau}\right\|_{C^{\alpha}(H)}\right).$$

Proof. First, we establish the estimate for $\left\|\{ABu_{k-1}\}_1^N\right\|_{C^\alpha(H)}$. By similar arguments given in the proof of estimate (25), we get

$$\left\|\{ABu_{k-1}\}_1^N\right\|_{C_\tau(H)} \le M(\delta,\theta_p)\left(\frac{1}{\alpha}\left\|\psi^\tau\right\|_{C^\alpha(H)} + \left\|A\varphi\right\|_H\right). \tag{38}$$

Let us now estimate

$$\max_{1\le k<k+r\le N} \frac{\left\|ABu_{k-1+r} - ABu_{k-1}\right\|_H}{(r\tau)^\alpha}.$$

From formula (13) it follows that for all $k=\overline{1,N}$

$$ABu_{k-1} = -\psi_k + D^{N-k+1}(BAu_N + \psi_N) - \sum_{j=k}^{N} \tau AD^{j-k}DB(\psi_{j-1} - \psi_k)$$
$$+D^{N-k}(\psi_k - \psi_N) = Q_1(k) + Q_2(k) + Q_3(k) + Q_4(k). \tag{39}$$

Clearly, we have

$$\left\|Q_1\right\|_{C^\alpha(H)} = \left\|\psi^\tau\right\|_{C^\alpha(H)}. \tag{40}$$

Next, we estimate $\left\|Q_2\right\|_{C^\alpha(H)}$. Let $\upsilon = (BAu_N + \psi_N)$. Note that

$$D^{N-k+1} - D^{N-k-r+1} = \sum_{m=0}^{r-1} -\tau A^{1-\alpha} D^{N-k-m} DBA^\alpha. \tag{41}$$

Thus, using the spectral theorem for A, the definition of H'_α, estimates (6) for $\alpha=0$, (9) for $\beta=1-\alpha$, and considering the cases $N-k\le 2r$ and $N-k>2r$ separately, we obtain

$$\left\|Q_2(k) - Q_2(k+r)\right\|_{(H)} \le \frac{4}{\alpha}\left\|\upsilon\right\|_{H'_\alpha}(r\tau)^\alpha. \tag{42}$$

Hence, it follows from (42) that

$$\|Q_2\|_{C^\alpha(H)} \le \frac{4}{\alpha}\|\upsilon\|_{H'_\alpha}. \tag{43}$$

Now, let us estimate $\|Q_3\|_{C^\alpha(H)}$. Using estimates (7) for $\alpha = 0$, (9) for $\beta = 1$, and the definition of $C^\alpha(H)$-norm that

$$\|Q_3(k)\|_H \le \frac{1}{\alpha}((N-k)\tau)^\alpha \|\psi^\tau\|_{C^\alpha(H)} \text{ for all } k. \tag{44}$$

Thus, it follows from estimate (44) that

$$\|Q_3\|_{C_\tau(H)} \le \frac{1}{\alpha}\|\psi^\tau\|_{C^\alpha(H)}. \tag{45}$$

We shall now estimate

$$\max_{1\le k<k+r\le N} \frac{\|Q_3(k+r)-Q_3(k)\|_H}{(r\tau)^\alpha}.$$

Let $N-k \le 2r$. Then by the triangle inequality, estimate (44), we obtain

$$\frac{\|Q_3(k+r)-Q_3(k)\|_H}{(r\tau)^\alpha} \le \frac{(2^\alpha+1)}{\alpha}\|\psi^\tau\|_{C^\alpha(H)}. \tag{46}$$

Next, let us consider the case $N-k > 2r$. Easily, we have

$$Q_3(k) - Q_3(k+r) = Q_{31}(k) + Q_{32}(k) + Q_{33}(k) + Q_{34}(k),$$

where $Q_{31}(k) = P_2(t), Q_{32}(k) = P_3(k), Q_{33}(k) = P_4(k)$ (see (27)) and

$$Q_{34}(k) = -(D^r - D^{N-k-r})(\psi_{k+r} - \psi_k). \tag{47}$$

$$\|Q_{32}(k)\|_H \le (r\tau)^\alpha \frac{M4^\alpha}{(1-\alpha)}\|\psi^\tau\|_{C^\alpha(H)},\ \|Q_{33}(k)\|_H \le (r\tau)^\alpha \frac{3^\alpha}{\alpha}\|\psi^\tau\|_{C^\alpha(H)}. \tag{48}$$

By estimate (6) for $\alpha = 0$ and the definition of $C^{\alpha}(H)$-norm, we obtain

$$\|Q_{34}(k)\|_H \leq 2(r\tau)^{\alpha} \|\psi^{\tau}\|_{C^{\alpha}(H)}. \tag{49}$$

From estimates (47)-(49) it follows that $N-k > 2r$

$$\frac{\|Q_3(t+\tau)-Q_3(t)\|_H}{(r\tau)^{\alpha}} \leq \frac{M}{\alpha(1-\alpha)} \|\psi^{\tau}\|_{C^{\alpha}(H)}. \tag{50}$$

Estimate (6) for $\alpha = 0$ and the definition of C^{α}-norm result that for all k

$$\|Q_4(k)\|_H \leq ((N-k)\tau)^{\alpha} \|\psi^{\tau}\|_{C^{\alpha}(H)} \leq \|\psi^{\tau}\|_{C^{\alpha}(H)}. \tag{51}$$

So, estimate (51) gives us

$$\|Q_4\|_{C_{\tau}(H)} \leq \|\psi^{\tau}\|_{C^{\alpha}(H)}. \tag{52}$$

It follows from estimates (6) for $\alpha = 0$, (10) for $\beta = 0,\ \ \gamma = 1,$ and definition of C^{α}-norm that for all $1 \leq k < k+r \leq N$

$$\|Q_4(k+r) - Q_4(k)\|_H \leq (M+1)(r\tau)^{\alpha} \|\psi^{\tau}\|_{C^{\alpha}(H)}. \tag{53}$$

Thus, using estimate (53), we obtain

$$\max_{1\leq k<k+r\leq N} \frac{\|Q_4(k+r)-Q_4(k)\|_H}{(r\tau)^{\alpha}} \leq M_1 \|\psi^{\tau}\|_{C^{\alpha}(H)}. \tag{54}$$

From estimates (52), (54) it result that

$$\|Q_4\|_{C^{\alpha}(H)} \leq M_1 \|\psi^{\tau}\|_{C^{\alpha}(H)}. \tag{55}$$

Combining estimates (40), (43), (50) and (55), we get

$$\left\|\left\{ABu_{k-1}\right\}_1^N\right\|_{C^\alpha(H)} \leq M\left(\frac{\left\|Au_N+\varphi_N\right\|_{H'_\alpha}}{\alpha}+\frac{\left\|\psi^\tau\right\|_{C^\alpha(H)}}{\alpha(1-\alpha)}\right). \tag{56}$$

From the triangle inequality, estimate (56) and difference equation (12) it follows

$$\left\|\{\tau^{-1}(u_k-u_{k-1})\}_1^N\right\|_{C^\alpha(H)} \leq M\left\{\frac{1}{\alpha}\left\|Au_N+\varphi_N\right\|_{H'_\alpha}+\frac{\left\|\psi^\tau\right\|_{C^\alpha(H)}}{\alpha(1-\alpha)}\right\}. \tag{57}$$

Let us now establish the estimate for $\left\|\{\tau^{-1}(u_k-u_{k-1})\}_1^N\right\|_{C(H'_\alpha)}$. Using difference equation (12) and formula (13), we obtain for all k

$$\begin{aligned}\tau^{-1}(u_k-u_{k-1}) &= D^{N-k+1}(BAu_N+\psi_N)+D^{N-k+1}(\psi_k-\psi_N)\\ &-B\sum_{j=k}^{N}\tau AD^{j-k+1}(\psi_j-\psi_k)=S_1(k)+S_2(k)+S_3(k).\end{aligned}$$

It follows from estimates (6) for $\alpha=0$, the definition of H'_α-norm that

$$\left\|S_1(k)\right\|_{H'_\alpha} \leq \left\|BAu_N+\psi_N\right\|_{H'_\alpha}. \tag{58}$$

By estimates (6) for $\alpha=0$, (9) for $\beta=\alpha$, and definition of H'_α-norm, we get

$$\left\|S_2(k)\right\|_{H'_\alpha} \leq M\left\|\psi^\tau\right\|_{C^\alpha(H)}. \tag{59}$$

Estimates (6) for $\alpha=0$, (9) for $\beta=1$, and definition of $C^\alpha(H)$-norm result

$$\left\|S_3(k)\right\|_H \leq \frac{1}{\alpha}\left\|\psi^\tau\right\|_{C^\alpha(H)}. \tag{60}$$

Using formula (37) and the definition of $C^{\alpha}(H)$-norm, we get

$$\left\|A^{\alpha}S_3(k)\right\|_H \le \frac{M(\delta)}{\alpha^2(1-\alpha)}\left\|\psi^{\tau}\right\|_{C^{\alpha}(H)}. \tag{61}$$

From estimates (60), (61) it follows that

$$\left\|S_3(k)\right\|_{H'_\alpha} \le \frac{M(\delta)}{\alpha^2(1-\alpha)}\left\|\psi^{\tau}\right\|_{C^{\alpha}(H)}. \tag{62}$$

Combining estimates (58)-(62), we obtain

$$\left\|\{\tau^{-1}(u_k-u_{k-1})\}_1^N\right\|_{C(H'_\alpha)} \le M\left\{\left\|BAu_N+\psi_N\right\|_{H'_\alpha}+\frac{\left\|\psi^{\tau}\right\|_{C^{\alpha}(H)}}{\alpha^2(1-\alpha)}\right\}. \tag{63}$$

Now, let us establish estimate $\left\|Au_N+\varphi_N\right\|_{H'_\alpha}$. From formula (14) it follows that

$$\begin{aligned}Au_N+\psi_N &= T_\tau\Bigg\{-\sum_{k=1}^{p}\alpha_k\,(I+d_kA)\sum_{j=\ell_k+1}^{N}\tau AD^{j-\ell_k}B(\psi_j-\psi_{\ell_k+1})\\ &\quad+\sum_{k=1}^{p}\alpha_k(I+d_kA)D^{N-\ell_k}(\psi_{\ell_k+1}-\psi_N)+\psi_N-\sum_{k=1}^{p}\alpha_k\psi_{\ell_k+1}+A\varphi\Bigg\}\\ &= U_1+U_2+U_3.\end{aligned}$$

Estimates (8), (10) for $\beta=1,(11)$, formula (37), assumption (2) and then definition $C^{\alpha}(H)$-norm give us

$$\left\|U_1\right\|_{H'_\alpha} \le \frac{M(\delta,\theta_p)}{\alpha^2(1-\alpha)}\left\|\psi^{\tau}\right\|_{C^{\alpha}(H)},\ \left\|U_2\right\|_{H'_\alpha} \le M(\delta,\theta_p)\left\|\psi^{\tau}\right\|_{C^{\alpha}(H)}, \tag{64}$$

$$\left\|U_3\right\|_{H'_\alpha} \le M(\delta,\theta_p)\left\|\psi_N-\sum_{k=1}^{p}\alpha_k\psi_{\ell_k+1}+A\varphi\right\|_{H'_\alpha}. \tag{65}$$

Thus, estimates (63)-(65) concludes the proof of Theorem 6.

3. AN APPLICATION

In this section, we present an application of Theorem 5 and Theorem 6. Let us consider the nonlocal boundary value problem

$$\begin{cases} u_t + (a(x)u_x)_x - \delta u = f(t,x), \ 0 < t < 1, \ 0 < x < 1, \\ u(1,x) = \sum_{m=1}^{p} \alpha_m u(\theta_m, x) + \varphi(x), \ 0 \le x \le 1, \\ 0 \le \theta_1 < \theta_2 < \cdots < \theta_p < 1, \\ u(t,0) = u(t,1), \ u_x(t,0) = u_x(t,1), \ 0 \le t \le 1 \end{cases} \tag{66}$$

under assumption (2), $\delta > 0$, $a(x) \ge a > 0$ $(x \in (0,1))$, $\varphi(x)$ $(x \in [0,1])$ and $f(t,x)$ $(t, x \in [0,1])$ are smooth functions.

Problem (66) is discretized in two steps. Let us first introduce the grid space

$$[0,1]_h = \{x = x_n : x_n = nh, \ \le n \le M, \ Mh = 1\}.$$

Let $L_{2h} = L_2([0,1]_h)$ be the Hilbert space of the grid functions $\varphi^h(x) = \{\varphi(x_n)\}_1^{M-1}$ defined on $[0,1]_h$, equipped with the norm

$$\left\| \varphi^h \right\|_{L_{2h}} = \left(\sum_{x \in [0,1]_h} |\varphi(x)|^2 h \right)^{1/2}.$$

To the differential operator A generated by problem (66) let us associate the difference operator A_h^x by the formula

$$A_h^x \varphi^h(x) = \{-(a(x)\varphi_{\bar{x}})_{x,n} + \delta \varphi_n\}_1^{M-1} \tag{67}$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi_n\}_1^{M-1}$ satisfying the conditions $\varphi_0 = \varphi_M$, $\varphi_1 - \varphi_0 = \varphi_M - \varphi_{M-1}$. It well-known that A_h^x is a self-adjoint positive definite operator in L_{2h}. Let $B_h^x = I + \frac{\tau A_h^x}{2}$. With the help of A_h^x, we arrive at the nonlocal boundary value problem

$$\begin{cases} \dfrac{du^h(t,x)}{dt} - A_h^x B_h^x u^h(t,x) = B_h^x f^h(t,x), \ t \in (0,1), \ x \in [0,1]_h, \\ u^h(1,x) = \sum_{m=1}^{p} \alpha_m u^h(\theta_m, x) + \varphi^h(x), \ x \in [0,1]_h. \end{cases} \tag{68}$$

In the second step, we replace (68) with difference scheme (3)

$$\begin{cases} \tau^{-1}(u_k^h(x) - u_{k-1}^h(x)) - A_h^x B_h^x u_{k-1}^h(x) = \psi_k(x), \\ \psi_k(x) = B_h^x f^h(t_{k-\frac{\tau}{2}}, x), \ t_k = k\tau, \ k = \overline{1,N}, \ N\tau = 1, \ x \in [0,1]_h, \\ u_N^h(x) = \sum\limits_{m=1}^{p} \alpha_m \{(I + d_m A_h^x) u_{\ell_m}^h(x) + d_m B_h^x \psi_{\ell_m+1}\} + \varphi^h(x), \ x \in [0,1]_h, \\ d_m = \theta_m - \left[\dfrac{\theta_m}{\tau}\right]\tau, \ \ell_m = \left[\dfrac{\theta_m}{\tau}\right] \ m = \overline{1,p}. \end{cases} \tag{69}$$

Theorem 7. Let τ and h be sufficiently small numbers. Then, the solutions of difference scheme (69) satisfy the following coercivity stability estimate

$$\left\| \{\tau^{-1}(u_k^h - u_{k-1}^h)\}_1^N \right\|_{C_1^\alpha([0,1]_\tau, L_{2h})} + \left\| \{u_{k-1}^h\}_1^N \right\|_{C_1^\alpha([0,1]_\tau, W_{2h}^2)}$$
$$\leq M(\delta, \theta_p) \left(\frac{1}{\alpha(1-\alpha)} \left\| \{\psi_k^h\}_1^N \right\|_{C_1^\alpha([0,1]_\tau, W_{2h}^2)} + \left\| \varphi^h \right\|_{W_{2h}^2} \right).$$

Theorem 8. Let $A_h^x \varphi^h(x) = \psi_N^h(x) \sum_{k=1}^{p} \alpha_k \psi_{\ell_k+1}^h(x)$. Then, for solutions of the problem (69), we have the following stability inequalities

$$\left\| \{\tau^{-1}(u_k^h - u_{k-1}^h)\}_1^N \right\|_{C^\alpha([0,1]_\tau, L_{2h})} + \left\| \{u_{k-1}^h\}_1^N \right\|_{C^\alpha([0,1]_\tau, W_{2h}^2)}$$
$$\leq \frac{M(\delta, \theta_p)}{\alpha^2(1-\alpha)} \left\| \{\psi_k^h\}_1^N \right\|_{C^\alpha(H)}.$$

The proof of Theorem 7, Theorem 8 is based on Theorem 5, Theorem 6 and the symmetry properties of the difference operator A_h^x defined by formula (67).

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